On the fundamentals of the three-dimensional translation gauge theory of dislocations

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Abstract

We propose a dynamic version of the three-dimensional translation gauge theory of dislocations. In our approach, we use the notions of the dislocation density and dislocation current tensors as translational field strengths and the corresponding response quantities (pseudomoment stress, dislocation momentum flux). We derive a closed system of field equations in a very elegant quasi-Maxwellian form as equations of motion for dislocations. In this framework, the dynamical Peach-Koehler force density is derived as well. Finally, the similarities and the differences between the Maxwell field theory and the dislocation gauge theory are presented.

Keywords: dislocation dynamics; gauge theory of dislocations; field theory; Peach-Koehler force.

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1 Introduction

In the last years, there has been a growing interest in continuum theories of dislocations. This development has been driven by the explanation of size effects in small-scale structures and by physically based plasticity theories of dislocations (see, e.g., [1]). On the other hand, it has been known for a long time that fast moving dislocations exhibit typical properties of moving particles and electromagnetic fields [2–4]. The stress field of a moving dislocation is longitudinally contracted. For that reason, the analogy between the theory of dislocations and the Maxwell theory of electromagnetic fields is often discussed in the literature [5–9]. Kröner [5] proposed an analogy between the dislocation theory and the theory of the magnetic field of distributions of stationary electric currents. Other authors [8, 9] suggested an analogy between the deformation and the magnetic fields and between the velocity and the electric fields. Also some authors [10, 11] used the analogy between the magnetic field and the dislocation density and between the electric field and the dislocation current density. Nevertheless, they did not use the concepts of excitations in dislocation theory which are necessary to obtain a complete theory with a closed system of equations of motion analogous to the electromagnetic field theory. Already, Schaefer [12] pointed out that constitutive equations are missing in the classical dislocation theory.

A dynamical theory of dislocations was formally derived in [13, 14]. We would like to mention that the Lagrangian of dislocations that they proposed contains only one material constant for the dislocation density tensor as well as for the dislocation current tensor. An improved and more realistic dislocation model has been formulated in [15–18]. In [18], the static solutions of screw and edge dislocations were given. The linear version of a dynamic dislocation gauge theory was formulated in [17] and successfully applied to a moving screw dislocation in the subsonic as well as in the supersonic regimes [19]. The nonuniformly motion of a dislocation has been investigated in [20]. Recently, Lazar and Hehl [21] have investigated Cartan's spiral staircase in the gauge theory of dislocations and they have shown that such a configuration arises naturally as a solution of the three-dimensional theory of dislocations.

The aim of this paper is to develop a dynamic theory of dislocations more systematically which makes use of the concepts of field strengths, excitations and constitutive relations. This theory is a kind of axiomatic field theory of dislocations. The nature of such a general dislocation theory is geometrically nonlinear. We are using the analogies to the axiomatic Maxwell theory given by Hehl and Obukhov [22]. The language of differential forms [23] is used as a powerful mathematical tool for overcoming structural decencies. Moreover, the framework of metric affine gauge theory (MAG) given by Hehl et al. [24] is also used in order to obtain a clear description of dislocation systems. The purpose of the paper is to give the differential geometric structure of the theory and not to discuss solutions of the field equations. For solutions of dislocations in the framework of dislocation gauge theory we refer to [16, 18–21].

2 Elastodynamics

In finite elasticity theory (see, e.g., [25–27]), the material body is identified with a three-dimensional manifold $\mathcal{M}^3$ which is embedded into the three-dimensional Euclidean space $\mathbb{R}^3$. We distinguish between the material or the final coordinates of $\mathcal{M}^3$, $a, b, c, \ldots = 1, 2, 3$, and the (holonomic) Cartesian coordinates of the reference system (defect-free or ideal reference system) $\mathbb{R}^3$, $i, j, k, \ldots = 1, 2, 3$. A *deformation* of $\mathbb{R}^3$ is a mapping $\boldsymbol{\xi} : \mathbb{R}^3 \longrightarrow \mathcal{M}^3$. A time-dependent family of deformation is called motion of $\mathcal{M}^3$. The *distortion* 1-form is defined by

$$\vartheta^a = B^a{}_i \, \mathrm{d}x^i = \mathrm{d}\xi^a \tag{1}$$

and is identified with a coframe. In elasticity, $B^a{}_i = \partial\xi^a/\partial x^i$ is called the *deformation gradient*, which is a so-called two-point tensor field because it is defined on two configurations or bases. ϑ^a and ξ^a have the dimensions: $[\vartheta^a] =$ length and $[\xi^a] =$ length. Here d is the three-dimensional exterior derivative. If the coframe (or distortion) 1-form has the property that

$$\mathrm{d}\vartheta^a = 0\,, \tag{2}$$

it said to be holonomic or compatible. Therefore, Eq. (2) is a compatibility condition for ϑ^a. The frame field

$$e_a = e_a{}^i \, \partial_i \tag{3}$$

is dual to the coframe (1) such that:

$$e_a \rfloor \vartheta^b = e_a{}^i \, B^b{}_i = \delta^b_a\,. \tag{4}$$

Here $\rfloor$ denotes the interior product. The frame e_a has the dimension: $[e_a] =$ 1/length.

A ϑ-basis for 0-, 1-, 2-, and 3-forms is $\{1, \vartheta^a, \vartheta^{ab} := \vartheta^a \wedge \vartheta^b, \vartheta^{abc} := \vartheta^a \wedge \vartheta^b \wedge \vartheta^c\}$, the Hodge dual η-basis for 3-,2-,1-, and 0-forms is specified by

$$\eta := {}^\star 1 = \frac{1}{3!}\, \eta_{abc}\, \vartheta^{abc}\,, \tag{5}$$

$$\eta_a := {}^\star\vartheta_a = e_a \rfloor \eta = \frac{1}{2}\, \eta_{abc}\, \vartheta^{bc}\,, \tag{6}$$

$$\eta_{ab} := {}^\star(\vartheta_{ab}) = e_b \rfloor \eta_a = \eta_{abc}\, \vartheta^c\,, \tag{7}$$

$$\eta_{abc} := {}^\star(\vartheta_{abc}) = e_c \rfloor \eta_{ab}\,, \tag{8}$$

where * denotes the Hodge star, $\wedge$ is the exterior product and $\eta_{abc} = \det\left(B^d{}_i\right) \epsilon_{abc}$, with ϵ_{abc} as the totally antisymmetric Levi-Civita symbol with $\pm 1, 0$.

Simultaneously, the *physical velocity* 0-form of the motion of the material continuum is given by

$$v^a = \frac{\partial \xi^a}{\partial t} \equiv \dot{\xi}^a. \tag{9}$$

It is a vector-valued 0-form. v^a describes the velocity of material points of the continuum and it has the dimension: $[v^a] = \text{length/time}$. The time-dependent distortion ϑ^a and the velocity field v^a have to satisfy the following kinematical compatibility condition:

$$\mathrm{d}v^a - \dot{\vartheta}^a = 0\,. \tag{10}$$

In elasticity, this is the kinematical compatibility condition between the deformation gradient and the associated velocity field.

The *right Cauchy-Green tensor* G is defined as the metric of the final state

$$G = g_{ab}\, \vartheta^a \otimes \vartheta^b = g_{ab}\, B^a{}_i B^b{}_j\, \mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{11}$$

If the coframe is orthonormal, then it reads

$$G = \delta_{ab}\, \vartheta^a \otimes \vartheta^b = \delta_{ab}\, B^a{}_i B^b{}_j\, \mathrm{d}x^i \otimes \mathrm{d}x^j \tag{12}$$

with $\delta_{ab} = \mathrm{diag}(+++)$. The *Lagrangian strain* tensor is given by

$$2E = G - 1 = (g_{ij} - \delta_{ij})\, \mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{13}$$

It measures the change of the metric between the undeformed and the deformed state.

The question of the response quantities (elastic excitations) is now of interest. This question is closely connected with the elastic field Lagrangian. In the continuum approach the elastic Lagrangian depends continuously on the elastic distortion and velocity. Thus, the *elastic Lagrangian* density 3-form is given by

$$\mathcal{L}_{\rm el} = \mathcal{L}_{\rm el}(v^a, \vartheta^a)\,. \tag{14}$$

As usual, the elastic Lagrangian density can be given in terms of kinetic and potential energy densities

$$\mathcal{L}_{\rm el} = T_{\rm k} - W\,, \tag{15}$$

where $T_{\rm k}$ is the *kinetic energy density* three-form and W denotes the *elastic potential energy density* 3-form (or distortion energy). The elastic potential energy density is a measure of the energy stored in the material as a result of elastic deformation. The excitation with respect to the physical velocity 0-form is the elastic *momentum* 3-form

$$p_a := \frac{\partial \mathcal{L}_{\rm el}}{\partial v^a}\,. \tag{16}$$

It is a covector valued 3-form

$$p_a = \frac{1}{3!}\, p_{aijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\,, \tag{17}$$

which has the dimension: $[p_a] = \text{momentum} \overset{\text{SI}}{=} \text{N s}$, and $[p_{aijk}] = \text{momentum}/(\text{length})^3$. The usual momentum vector (covector valued 0-form), known form elasticity theory, is given by: $P_a = (1/3!)\eta^{ijk}p_{aijk}$. In elasticity, the linear and isotropic constitutive law for the momentum vector is of the form: $P_a = \rho g_{ab} v^b$, where ρ denotes the mass density.

The elastic *force stress* 2-form is the excitation quantity relative to the distortion 1-form and is defined by

$$\Sigma_a := \frac{\delta \mathcal{L}_{\text{el}}}{\delta \vartheta^a} = \frac{\partial \mathcal{L}_{\text{el}}}{\partial \vartheta^a}\,. \tag{18}$$

Eq. (18) is the general constitutive relation for nonlinear elasticity. Σ_a is a covector valued 2-form

$$\Sigma_a = \frac{1}{2}\,\Sigma_{aij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j\,. \tag{19}$$

It has the dimension: $[\Sigma_a] = \text{force} \overset{\text{SI}}{=} \text{N}$, and for the components: $[\Sigma_{aij}] = \text{force}/(\text{length})^2 = \text{stress} \overset{\text{SI}}{=} \text{P}$. We recognize that the force stress has 9 independent components. Therefore, the first Piola-Kirchhoff stress tensor (a two-point tensor) is represented by: ${\Sigma_a}^k = (1/2)\eta^{ijk}\Sigma_{aij}$. In elasticity theory, the first Piola-Kirchhoff stress tensor reads: ${\Sigma_a}^k = e_{aj}\sigma^{jk}$, where the Cauchy stress tensor is given by the generalized Hooke law $\sigma^{jk} = C^{jkmn}E_{mn}$ with the elasticity tensor C^{jkmn}.

In elastodynamics, the Euler-Lagrange equations give the following equation of motion

$$\frac{\delta \mathcal{L}_{\text{el}}}{\delta \xi^a} \equiv \frac{\partial \mathcal{L}_{\text{el}}}{\partial \xi^a} - \mathrm{d}\,\frac{\partial \mathcal{L}_{\text{el}}}{\partial \mathrm{d}\xi^a} - \partial_t\,\frac{\partial \mathcal{L}_{\text{el}}}{\partial \dot{\xi}^a} = 0 \tag{20}$$

and in terms of the stress 2-form and momentum 3-form they are of the form

$$\dot{p}_a + \mathrm{d}\Sigma_a = 0\,. \tag{21}$$

The *elastic energy* density $\mathcal{E}_{\text{el}}$ is defined in the framework of field theory as the Hamiltonian of the elastic system

$$\mathcal{E}_{\text{el}} := p_a v^a - \mathcal{L}_{\text{el}} = T_{\text{k}} + W. \tag{22}$$

3 T(3)-gauge theory of dislocations

In this section, we discuss the three-dimensional translation gauge theory of dislocations. In this approach, we consider the translation group, $T(3)$, as a gauge group. We assume that all fields depend on the space and time variables. The $T(3)$-transformation acts on ξ^a as a gauge transformation in the following way

$$\xi^a \longrightarrow \xi^a - \tau^a(x,t)\,, \tag{23}$$

where $\tau^a(x,t)$ are local and time-dependent translations. If we do so, the invariance of the compatible distortion (1) and the material velocity (9) are lost under the local $T(3)$-transformations (23). In order to compensate the invariance violating terms, we have to introduce gauge potentials, a vector-valued 1-form $\phi^a = \phi^a{}_i\, \mathrm{d}x^i$ and a vector valued 0-form φ^a transforming under the local transformations in a suitable way:

$$\phi^a \longrightarrow \phi^a + \mathrm{d}\tau^a(x,t) \tag{24}$$

$$\varphi^a \longrightarrow \varphi^a + \dot{\tau}^a(x,t)\,. \tag{25}$$

Thus, ϕ^a and φ^a are the translational gauge potentials of the dynamical $T(3)$-gauge theory. Since the elastic distortion and the material velocity are state quantities in the field theory of dislocations, they have to be gauge-invariant. Now we redefine the elastic distortion and the material velocity in a gauge-invariant form as follows:

$$\vartheta^a := \mathrm{d}\xi^a + \phi^a \tag{26}$$

$$v^a := \dot{\xi}^a + \varphi^a\,. \tag{27}$$

Hence, ξ^a, ϕ^a and φ^a appear in (26) and (27) always joined together in the translation-invariant combinations. The gauge fields ϕ^a and φ^a make the elastic distortion ϑ^a and the material velocity v^a incompatible because they are not any longer just a simple gradient and a time-derivative of ξ^a.

In [15] we have seen that ϕ^a in Eq. (26) can be interpreted as the translational part of the generalized affine connection in a Weitzenböck space. The underlying geometrical structure of the theory is the affine tangent bundle AM. At every point, the tangent space is replaced by the affine tangent space. The translation group $T(3)$ acts on the affine space as an internal symmetry. The field ξ^a is also known as Cartan's 'radius vector' and determines the 'origin' of the affine space [24]. In the context of dynamical $T(3)$-gauge theory, two translational gauge potentials ϕ^a and φ^a and one translational Goldstone field ξ^a occur and they are the canonical field quantities. On the other hand, in the 'classical' theory of defects (see, e.g., [28]) fields like ϕ^a and φ^a are called the negative plastic (or initial) distortion and velocity, respectively.

3.1 Translational field strengths

Because we have two translational gauge potentials ϕ^a and φ^a, we may define two translational field strengths in terms of these gauge potentials. We introduce the well-known quantities of dislocation density and dislocation current as field strengths of the $T(3)$-gauge theory breaking the compatibility conditions (2) and (10). The *dislocation density* 2-form T^a is defined as the *object of anholonomity* or *torsion* 2-form in a teleparallel space (Weitzenböck space). The torsion 2-form (dislocation density 2-form) is defined in terms of the translational gauge

potential ϕ^a

$$T^a = \mathrm{d}\phi^a \tag{28}$$

and, alternatively, in terms of the anholonomic coframe

$$T^a = \mathrm{d}\vartheta^a \,. \tag{29}$$

Thus, T^a measures how much the gauge potential ϕ^a and the coframe ϑ^a fail to be holonomic or compatible. The torsion is a vector valued 2-form

$$T^a = \frac{1}{2}\, T^a{}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \,, \tag{30}$$

which has the dimension: $[T^a]$ = length. The torsion tensor $T^a{}_{ij}$ has the dimension: $[T^a{}_{ij}] = 1/\text{length}$. The field $T^a{}_{12}$ measures the number of dislocation lines going in 3-direction and having the Burgers vector b^a. T^a can describe a continuous distribution of dislocations as well as single dislocations. The usual dislocation density tensor [6] is represented by: $\alpha_a{}^k = (1/2)\eta^{ijk} T^a{}_{ij}$.

Another translational field strength is the *dislocation current*. The dislocation current 1-form is defined in terms of the translational gauge potentials φ^a and ϕ^a as follows

$$I^a = \mathrm{d}\varphi^a - \dot{\phi}^a \tag{31}$$

or in terms of the incompatible coframe and incompatible velocity as

$$I^a = \mathrm{d}v^a - \dot{\vartheta}^a \,. \tag{32}$$

In Eq. (32), we see that the dislocation current is defined in terms of the physical velocity gradient and the rate of the elastic distortion. It is a vector valued 1-form

$$I^a = I^a{}_i \, \mathrm{d}x^i \,, \tag{33}$$

with the dimension: $[I^a]$ = velocity $\overset{\text{SI}}{=}$ m/s. The dislocation current tensor has the dimension: $[I^a{}_i] = 1/\text{time} \overset{\text{SI}}{=} 1/\text{s}$. The translational field strengths T^a and I^a measure the violation of the two compatibility conditions (2) and (10) and, therefore, how much the elastic distortion ϑ^a and the physical velocity v^a are incompatible. The dislocation density T^a and the dislocation current I^a are state quantities and they can be observed experimentally. The dislocation current is the appropriate quantity for the description of dynamics of dislocations. For the dynamical case, I^a and T^a carry the information about the dislocation state of motion.

The two translational field strengths have to satisfy the following Bianchi identities:

$$\mathrm{d}T^a = 0 \,, \tag{34}$$

$$\mathrm{d}I^a + \dot{T}^a = 0 \,. \tag{35}$$

Eq. (34) is the well-known conservation law of dislocations (continuity equation of the dislocation density) and (35) is known as the continuity equation of the dislocation current [29, 30]. Eq. (34) states that a dislocation line cannot end inside the body. The evolution of T^a is determined by I^a in a closed form. If we integrate Eq. (34) over a three-dimensional volume which contains dislocations and use the Stokes theorem, we obtain the Burgers vector by means of the Burgers circuit ∂S

$$\int_S T^a = \oint_{\partial S} \vartheta^a = b^a \tag{36}$$

where ∂S is the boundary of the surface S. On the other hand, if we integrate Eq. (35) over a two-dimensional surface S and we use the Stokes theorem, we get the so-called 'conservation law of the Burgers vector'

$$\oint_{\partial S} I^a + \dot{b}^a = 0\,. \tag{37}$$

The integral of Eq. (37) determines the flux of the Burgers vector b^a per unit time through the contour ∂S. It states that the time change of the Burgers vector on a surface S is equal to the negative dislocation current over the contour ∂S of the surface S.

If the local translation is not time-dependent $\tau^a = \tau^a(x)$, then the physical velocity v^a is compatible like in (9) and one may use the gauge condition $\varphi^a = 0$ in order to eliminate the gauge potential φ^a. Such a gauge condition is called temporal gauge (or Weyl gauge) in gauge field theories. In this gauge, the dislocation current (31) reads: $I^a = -\dot{\phi}^a$. Such a situation is used, e.g., in [29, 30] if we identify the gauge potential ϕ^a with the negative plastic distortion. In this way, the dislocation current is related to the rate of plastic distortion.

3.2 Gauge field momenta – dislocation field excitations

To complete the field theory of dislocations, we have to define the excitations with respect to the dislocation density and the dislocation current, respectively. The dislocation Lagrangian density is of the form:

$$\mathcal{L}_{\rm disl} = \mathcal{L}_{\rm disl}(v^a, \vartheta^a, I^a, T^a)\,. \tag{38}$$

Temporarily we will leave open the explicit form of $\mathcal{L}_{\rm disl}$. Then it is necessary to introduce the field momenta that are canonically conjugated to the translational field strengths. These field momenta are called the dislocation excitations. The excitation with respect to the torsion 2-form is defined by

$$H_a := -\frac{\partial \mathcal{L}_{\rm disl}}{\partial T^a}\,. \tag{39}$$

It is the specific response to T^a. Another excitation is the 2-form D_a

$$D_a := \frac{\partial \mathcal{L}_{\rm disl}}{\partial I^a}\,. \tag{40}$$

It is the response of the dislocation Lagrangian to I^a. We may interpret H_a and D_a as the *pseudomoment stress* and the *dislocation momentum flux*, respectively, caused by the motion of dislocations. Here, H_a is a covector valued 1-form

$$H_a = H_{ai}\,{\rm d}x^i\,. \tag{41}$$

The dimension of H_a is: $[H_a]$ = force, and for its components: $[H_{ai}]$ = force/length. D_a is a covector valued 2-form

$$D_a = \frac{1}{2}\,D_{aij}\,{\rm d}x^i \wedge {\rm d}x^j\,, \tag{42}$$

which has the dimension: $[D_a]$ = momentum, and $[D_{aij}]$ = momentum/(length)^2. Eqs. (39) and (40) are constitutive relations of the nonlinear dislocation field theory.

In addition, we define the *dislocation stress* as a covector valued 2-form as

$$E_a := \frac{\partial \mathcal{L}_{\rm disl}}{\partial \vartheta^a} \tag{43}$$

with the dimension $[E_a]$ = force, and the *dislocation momentum* 3-form reads

$$\pi_a := \frac{\partial \mathcal{L}_{\rm disl}}{\partial \upsilon^a}\,, \tag{44}$$

which is a covector valued 3-form and it has the dimension: $[\pi_a]$ = momentum. Eqs. (43) and (44) are valid for the dislocation stress and the dislocation momentum in the nonlinear dislocation field theory. The dislocation stress 2-form is explicitely given by

$$E_a = e_a\rfloor\mathcal{L}_{\rm disl} + (e_a\rfloor T^b)\wedge H_b - (e_a\rfloor I^b)D_b \tag{45}$$

and the dislocation momentum 3-form reads

$$\pi_a = (e_a\rfloor T^b)\wedge D_b = -(e_a\rfloor D_b)\wedge T^b\,. \tag{46}$$

In continuum mechanics, (45) and (46) are called the Eshelby stress and the pseudomomentum (see, e.g., [27, 31]).

3.3 Dislocation field equations

We are now ready to derive the Yang-Mills type field equations determining the dynamics of dislocations. The total Lagrangian density is of the form

$$\mathcal{L} = \mathcal{L}_{\rm disl} + \mathcal{L}_{\rm el}\,. \tag{47}$$

The variation of the total Lagrangian with respect to the Goldstone field ξ^a and the translational gauge potentials φ^a and ϕ^a gives the Euler-Lagrange equations. According to the extremal action principle, the field equations are found to be

$$\frac{\delta\mathcal{L}}{\delta\xi^a} \equiv \frac{\partial\mathcal{L}}{\partial\xi^a} - \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial\mathrm{d}\xi^a} - \partial_t\,\frac{\partial\mathcal{L}}{\partial\dot{\xi}^a} = 0 \tag{48}$$

$$\frac{\delta\mathcal{L}}{\delta\varphi^a} \equiv \frac{\partial\mathcal{L}}{\partial\varphi^a} - \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial\mathrm{d}\varphi^a} - \partial_t\,\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^a} = 0 \tag{49}$$

$$\frac{\delta\mathcal{L}}{\delta\phi^a} \equiv \frac{\partial\mathcal{L}}{\partial\phi^a} + \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial\mathrm{d}\phi^a} - \partial_t\,\frac{\partial\mathcal{L}}{\partial\dot{\phi}^a} = 0\,. \tag{50}$$

Alternatively, we may variate the total Lagrangian with respect to the gauge-invariant quantities, namely the incompatible velocity v^a and the coframe ϑ^a:

$$\frac{\delta\mathcal{L}}{\delta v^a} \equiv \frac{\partial\mathcal{L}}{\partial v^a} - \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial\mathrm{d}v^a} - \partial_t\,\frac{\partial\mathcal{L}}{\partial\dot{v}^a} = 0 \tag{51}$$

$$\frac{\delta\mathcal{L}}{\delta\vartheta^a} \equiv \frac{\partial\mathcal{L}}{\partial\vartheta^a} + \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial\mathrm{d}\vartheta^a} - \partial_t\,\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}^a} = 0\,, \tag{52}$$

which can be expressed in the following form

$$\frac{\delta\mathcal{L}}{\delta v^a} \equiv \frac{\partial\mathcal{L}}{\partial v^a} - \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial I^a} = 0 \tag{53}$$

$$\frac{\delta\mathcal{L}}{\delta\vartheta^a} \equiv \frac{\partial\mathcal{L}}{\partial\vartheta^a} + \mathrm{d}\,\frac{\partial\mathcal{L}}{\partial T^a} + \partial_t\,\frac{\partial\mathcal{L}}{\partial I^a} = 0\,. \tag{54}$$

These field equations of dislocation theory (53) and (54) may be expressed in terms of the response quantities (39), (40), (43) and (44) in order to take the following from

$$\mathrm{d}D_a - \pi_a = p_a \tag{55}$$

$$\mathrm{d}H_a - \dot{D}_a - E_a = \Sigma_a\,. \tag{56}$$

Eqs. (55) and (56) are two Yang-Mills type field equations with the momentum $(p_a + \pi_a)$ and the force stress $(\Sigma_a + E_a)$ as sources of the dislocation excitations D_a and H_a, respectively. The translational gauge fields themselves cause 'gauge' sources π_a and E_a, thereby contributing to their own elastic sources p_a and Σ_a. Due to the complexity of the dislocation gauge field interaction, there arise self-couplings which involve the dislocation momentum π_a and the dislocation stress E_a. Thus, the field equations (55) and (56) are nonlinear. In addition, Eqs. (55) and (56) constitute a closed system of 12 independent field equations for the state quantities v^a and ϑ^a. Eq. (55) can be interpreted as the equilibrium equation between the dislocation momentum flux and the momenta (balance equation of momenta). In the static case, Eq. (55) is vanishing. Eq. (56) is the equilibrium equation between the pseudomoment stress, dislocation momentum flux and force

stresses (balance equation of stresses). From Eqs. (55) and (56), we obtain the following conservation law:

$$\dot{p}_a + \dot{\pi}_a + \mathrm{d}\big(\Sigma_a + E_a\big) = 0\,. \tag{57}$$

Eq. (57) is nothing but the Euler-Lagrange equation (48) which is the force equilibrium condition if dislocations are present (continuity equation of force stresses). It determines the exchange of momentum and stress between the elastic and the dislocation subsystems. Observe that, in contrast to standard elasticity theory, the conserved quantities are the total momentum and the total force stress of the system and not the elastic quantities themselves. Linear solutions of the field equations (55) and (56) for moving dislocations are given by Lazar [19, 20].

In order to complete the framework of dislocation field theory, we introduce the *Peach-Koehler force* 3-form

$$f_a := -\dot{\pi}_a - \mathrm{d}E_a = \dot{p}_a + \mathrm{d}\Sigma_a = \big(e_a \rfloor I^b\big) p_b + \big(e_a \rfloor T^b\big) \wedge \Sigma_b\,. \tag{58}$$

It represents the force density acting on dislocations. Eq. (58) is the dynamical form of the Peach-Kohler force which is analogous to the Lorentz force [22] in Maxwell's theory of electromagnetic fields. The force density 3-form is a covector valued 3-form according to

$$f_a = \frac{1}{3!}\, f_{aijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{59}$$

and the dimension of the force density tensor is: $[f_{aijk}] = \text{force}/(\text{length})^3$.

The *moment stress* 2-form τ_{ab} is related to the excitation H_a according to

$$\tau_{ab} := \vartheta_{[a} \wedge H_{b]} \tag{60}$$

with the dimension: $[\tau_{ab}] = \text{force} \times \text{length}$. In components, the moment stress 2-form reads

$$\tau_{ab} = \frac{1}{2}\, \tau_{abij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \tag{61}$$

and the dimension of the moment stress tensor is: $[\tau_{abij}] = \text{force}/\text{length}$. The formula (60) can be inverted as follows:

$$H_a = -2 e_b \rfloor \tau_a{}^b + \frac{1}{2}\, \vartheta_a \wedge \left(e_b \rfloor e_c \rfloor \tau^{bc}\right)\,. \tag{62}$$

In order to derive the *moment equilibrium*, it requires some algebra. We start with the field equation (56) and compute the antisymmetric piece of the total stress, use (60), and find finally

$$\mathrm{d}\tau_{ab} - T_{[a} \wedge H_{b]} + \vartheta_{[a} \wedge \dot{D}_{b]} + \vartheta_{[a} \wedge \big(E_{b]} + \Sigma_{b]}\big) = 0\,. \tag{63}$$

Apart from the terms $-T_{[a} \wedge H_{b]}$ and $\vartheta_{[a} \wedge \dot{D}_{b]}$, this is exactly the expected law known from continuum mechanics.

3.4 Quadratic gauge field Lagrangian

Now we may specify *constitutive laws.* In order to give concrete expressions for the excitations, we have to specify the constitutive relations between field strengths (T^a, I^a) and excitations (H_a, D_a). For a local, linear, isotropic continuum we have

$$H_a = g_{ab}\,{}^\star\sum_{I=1}^{3} a_I\,{}^{(I)}T^b\,, \tag{64}$$

wherein ${}^{(I)}T_a$ are the irreducible pieces (65), (66), and (67) of the torsion and a_1, a_2, and a_3 are constitutive moduli which have the dimension: $[a_I] =$ force. ${}^\star T^a$ is dimensionless. We may decompose the torsion into three $SO(3)$-irreducible pieces according to $T^a = {}^{(1)}T^a + {}^{(2)}T^a + {}^{(3)}T^a$ with the number of independent components $9 = 5\oplus3\oplus1$. These three pieces (tensor piece, trace- and axial-vector pieces of the torsion) are defined by

$$\begin{aligned}
{}^{(1)}T^a &:= T^a - {}^{(2)}T^a - {}^{(3)}T^a && \text{(tentor)}, && (65)\\
{}^{(2)}T^a &:= \frac{1}{2}\,\vartheta^a \wedge \big(e_b\rfloor T^b\big) && \text{(trator)}, && (66)\\
{}^{(3)}T_a &:= \frac{1}{3}\,e_a\rfloor\big(\vartheta^b \wedge T_b\big) && \text{(axitor)}\,. && (67)
\end{aligned}$$

In addition, we have for the local, linear, isotropic continuum the constitutive relation between D_a and I^a:

$$D_a = g_{ab}\,{}^\star\sum_{I=1}^{3} f_I\,{}^{(I)}I^b\,, \tag{68}$$

wherein ${}^{(I)}I_a$ are the irreducible pieces (69), (70), and (71) of the dislocation current and $[{}^\star I^a] = (\text{length})^2/\text{time}$. f_1, f_2, and f_3 are constitutive moduli which have the dimension: $[f_I] = \text{mass/length} \overset{\text{SI}}{=} \text{kg/m}$. The three $SO(3)$-irreducible pieces $I^a = {}^{(1)}I^a + {}^{(2)}I^a + {}^{(3)}I^a$ with the number of independent components $9 = 5 \oplus 3 \oplus 1$ are given by

$$\begin{aligned}
{}^{(1)}I^a &:= I^a - {}^{(2)}I^a - {}^{(3)}I^a && \text{(symmetric and traceless)}, && (69)\\
{}^{(2)}I_a &:= \frac{1}{2}\,e_a\rfloor\big(\vartheta^b \wedge I_b\big) && \text{(antisymmetric)}, && (70)\\
{}^{(3)}I^a &:= \frac{1}{3}\,\vartheta^a\big(e_b\rfloor I^b\big) && \text{(trace)}\,. && (71)
\end{aligned}$$

Thus, for a linear continuum, the dislocation Lagrangian has the bilinear form

$$\mathcal{L}_{\text{disl}} = \frac{1}{2}\,I^a \wedge D_a - \frac{1}{2}\,T^a \wedge H_a\,. \tag{72}$$

The pure *dislocation energy* is defined as the Hamiltonian of the dislocation system

$$\mathcal{E}_{\rm disl} := I^a \wedge D_a - \mathcal{L}_{\rm disl} = \frac{1}{2}\, I^a \wedge D_a + \frac{1}{2}\, T^a \wedge H_a \,. \tag{73}$$

More physically, we can interpret $\mathcal{E}_{\rm disl}$ as the dislocation core energy. Moreover, we see in Eqs. (72) and (73) that the first term plays the role of the kinetic dislocation energy and the second one can be identified with the potential energy of dislocations. Thus, the excitation D_a is a kind of 'momentum' and I^a plays the role of a generalized 'velocity' of the dislocation motion.

4 Discussion and Conclusion

Based on the $T(3)$-gauge theory, we have proposed a dynamical field theory of dislocations. We have used the concepts of field strengths, excitations and constitutive laws analogical to the electromagnetic field theory in order to obtain a closed field theory. All dislocation field quantities can be described by $\mathbb{R}^3$-valued exterior differential forms. The translation field strengths are even (or polar) differential forms and the excitations (stresses and momenta) are odd (or axial) forms. We have shown that the excitations to the dislocation density and dislocation current are necessary for a realistic physical dislocation field theory. Moreover, we have demonstrated how the excitations have to fit into the Maxwell-type field equations in contrast to [10] who claimed that, there are no analogues to the second pair of the Maxwell equations in dislocation theory. A review of the corresponding electromagnetic and dislocation quantities is given in Table 1. The gauge theory of dislocations is a closed field theory. However, there are important distinctions. In the Maxwell theory the field quantities are scalar-valued forms and in the dislocation field theory all field quantities are vector-valued (or covector-valued) forms. Due to self-couplings the field equations for dislocations are nonlinear. The electromagnetic current j depends on an exterior material field, while the stress $(E_a+\Sigma_a)$ is interior and it depends on the coframe (distortion) itself.

It is known that dislocations in crystals move in two different modes, called glide (conservative motion) and climb (non-conservative motion). For instance, the volume and the mass density, respectively, of the crystal is not changed by the gliding of dislocations. On the other hand, climbing dislocations interchange with point defects such as vacancies and/or interstitials. Additionally, if dislocations cut each other, they build networks of dislocations. In our dynamical dislocation theory, we have neglected dissipation (friction and radiation damping) and the interaction with point defects. In order to take into account the energy dissipated and converted into heat one can use a Lagrangian extended by a dissipation function (see, e.g., [32]).

To sum up, the state quantities in dislocation dynamics are the physical velocity v^a, the elastic distortion ϑ^a, the dislocation density T^a and the dislocation

Maxwell field theory	Dislocation field theory
B - magnetic field strength	T^a - dislocation density
E - electric field strength	I^a - dislocation current
H - magnetic excitation	H_a - pseudomoment stress
D - electric excitation	D_a - dislocation momentum flux
A - magnetic potential 1-form	ϕ^a - dislocation potential 1-form
φ - potential 0-form	φ^a - dislocation potential 0-form
f - gauge function	ξ^a - deformation mapping
$B = \mathrm{d}A, \quad A' = A + \mathrm{d}f$	$T^a = \mathrm{d}\phi^a, \quad \phi^{a\prime} \equiv \vartheta^a = \phi^a + \mathrm{d}\xi^a$
$E = \mathrm{d}\varphi - \dot{A}, \quad \varphi' = \varphi + \dot{f}$	$I^a = \mathrm{d}\varphi^a - \dot{\phi}^a, \quad \varphi^{a\prime} \equiv v^a = \varphi^a + \dot{\xi}^a$
Φ - magnetic flux of magnetic vortices	b^a - Burgers vector of dislocations
$\int_S B = \Phi$	$\int_S T^a = b^a$
ρ - electric charge density	$p^{\mathrm{T}}_a = p_a + \pi_a$ - total momentum density
j - electric current	$\Sigma^{\mathrm{T}}_a = \Sigma_a + E_a$ - total force stress
magnetic field closed:	continuity equation of dislocation density:
$\mathrm{d}B = 0$	$\mathrm{d}T^a = 0$
Faraday law:	continuity equation of dislocation current:
$\mathrm{d}E + \dot{B} = 0$	$\mathrm{d}I^a + \dot{T}^a = 0$
Gauss law:	continuity equation of dislocation moment flux:
$\mathrm{d}D = \rho$	$\mathrm{d}D_a = p^{\mathrm{T}}_a$
Oersted-Ampère law:	continuity equation of moment stress:
$\mathrm{d}H - \dot{D} = j$	$\mathrm{d}H_a - \dot{D}_a = \Sigma^{\mathrm{T}}_a$
continuity equation of current:	continuity equation of force stress:
$\dot{\rho} + \mathrm{d}j = 0$	$\dot{p}^{\mathrm{T}}_a + \mathrm{d}\Sigma^{\mathrm{T}}_a = 0$
constitutive laws:	constitutive laws:
$H = H(B)$, $D = D(E)$	$H_a = H_a(T^b)$, $D_a = D_a(I^b)$
Lorentz force density:	Peach-Koehler force density:
$f_a = (e_a \rfloor E)\rho + (e_a \rfloor B) \wedge j$	$f_a = (e_a \rfloor I^b)p_b + (e_a \rfloor T^b) \wedge \Sigma_b$

Table 1: The correspondence between Maxwell's theory and dislocation field theory.

current I^a. In a dislocation field theory based on these state quantities one finds the Euler-Lagrange equations and the response quantities as we have given in this paper. In addition, one can combine such a dislocation theory with the so-called 'multiplicative decomposition' [33–35] which is widely used and accepted in engineering science. Nevertheless, the geometrical or field theoretical arena of a dislocation field theory is the gauge theory of the three-dimensional translation group. It is just a consequence from the fact that dislocations break locally the translation symmetry in the crystal and that the dislocation density tensor is nothing but a realization of Cartan's torsion tensor [36, 37] in three dimensions what was originally found by Kondo [38] (see also [34]).

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